Lecture 6 on Sept. 26

In the lecture today, we begin to study the so-called rational functions.

Definition 0.1. *Given two polynomials $P(z)$ and $Q(z)$,*

$$R(z) = \frac{P(z)}{Q(z)}$$

is called a rational function.

Here are some remarks that we should have to know

Remark 0.2. *If $R_1(z)$ and $R_2(z)$ are two rational functions, then $R_1 \pm R_2$, $R_1 R_2$, R_1/R_2 are also rational functions. Moreover if we assume $R(z) = P(z)/Q(z)$, then by quotient rule*

$$R'(z) = \frac{QP' - PQ'}{Q^2},$$

which is still a rational function.

Remark 0.3. *By the fundamental theorem of algebra, we can factorize $P(z)$ and $Q(z)$ as follows*

$$P(z) = c_1 \Pi_{k=1}^m (z - \alpha_k), \qquad Q(z) = c_2 \Pi_{j=1}^n (z - \beta_j).$$

Therefore

$$R(z) = \frac{P(z)}{Q(z)} = \frac{c_1}{c_2} \frac{\Pi_{k=1}^m (z - \alpha_k)}{\Pi_{j=1}^n (z - \beta_j)}. \tag{0.1}$$

Applying cancellation to the above quotient, we know that we can always assume α_k are different from β_j. In other words

$$\{\alpha_k : k = 1, ..., m\} \bigcap \{\beta_j : j = 1, ..., n\} = \emptyset.$$

before proceeding, let us study how to write a rational function into the sum of partial fractions. One should only follow the guideline below.

Step 1. Assuming $R(z) = P(z)/Q(z)$, we use long division to rewrite $R(z)$ as

$$R(z) = G(z) + H(z). \tag{0.2}$$

here $G(z)$ is a polynomial while $H(z)$ is a real (proper) rational function. Here we mean a rational function proper if the order of the nominator polynomial is smaller than the order of the denominator polynomial. Notice that if H is proper then $H(\infty) = 0$;

Step 2. Assuming β_1 ... β_n are n distinct roots of the polynomial Q, we consider the rational function $H(\beta_j + \frac{1}{w})$ for each $j = 1, ..., n$. If $H(z)$ is proper in terms of variable z, then $H(\beta_j + \frac{1}{w})$ must not be proper in terms of variable w. Therefore we can do long division for $H(\beta_j + \frac{1}{w})$ and show that

$$H(\beta_j + \frac{1}{w}) = G_j(w) + H_j(w). \tag{0.3}$$

where G_j is a polynomial of w and H_j is a proper rational function. Using this G_j and G from the first step, we can write

$$R(z) = G(z) + \sum_j G_j(\frac{1}{z - \beta_j}) + C,$$

where C is a constant. This is the so-called sum of partial fractions for $R(z)$.

Step 3. Now we determine the constant C. Since G_j in Step 2 is a polynomial, then it has a constant term denoted by $C(G_j)$. C equals to $-\sum_j C(G_j)$.

In fact, by Remark 0.2, we know that

$$\tilde{R}(z) = R(z) - G(z) - \sum_j G_j(\frac{1}{z-\beta_j})$$

must be a rational function. when $z \neq \beta_j$ for all $j = 1, ..., n$, then $R(z)$,$G(z)$ and $G_j(1/(z-\beta_j))$ are all finite complex numbers. Therefore $\tilde{R}(z)$ is finite. Letting β be one of complex numbers in $\{\beta_j\}$, then by (0.2) we know that

$$\tilde{R}(\beta) = H(\beta) - G_{j^*}(\frac{1}{\beta-\beta_{j^*}}) - \sum_{j \text{ such that } \beta_j \neq \beta} G_j(\frac{1}{\beta-\beta_j}) \tag{0.4}$$

Here j^* is the index such that $\beta_{j^*} = \beta$. Supposing that $z = \beta_{j^*} + 1/w$, by (0.3), we have

$$H(z) = G_{j^*}(\frac{1}{z-\beta_{j^*}}) + H_{j^*}(\frac{1}{z-\beta_{j^*}})$$

Clearly it holds

$$H(\beta) - G_{j^*}(\frac{1}{\beta-\beta_{j^*}}) = H_{j^*}(\frac{1}{\beta-\beta_{j^*}}).$$

Since H_{j^*} is proper, therefore we have

$$H_{j^*}(\frac{1}{\beta-\beta_{j^*}}) = H_{j^*}(\infty) = 0.$$

Applying the above two equalities to (0.4), we know that $\tilde{R}$ is also finite at β. Therefore $\tilde{R}$ is a rational function such that $\tilde{R}(z)$ is finite for all z in $\mathbb{C}$. Such rational function can only be a polynomial. Furthermore by the fact that

$$\tilde{R}(z) = H(z) - \sum_j G_j(\frac{1}{z-\beta_j}),$$

we know that at ∞,

$$\tilde{R}(\infty) = H(\infty) - \sum_j G_j(0) = -\sum_j G_j(0).$$

Here the fact that H is proper is used. The above arguments show that $\tilde{R}$ is a polynomial and must be finite at ∞. Such polynomial can only be a constant. Up to now we have shown that

$$R(z) = G(z) + \sum_j G_j(\frac{1}{z-\beta_j}) + C.$$

Applying (0.2) to the above equality, we know that

$$H(z) = \sum_j G_j(\frac{1}{z-\beta_j}) + C.$$

Therefore by the fact that H is proper, we have

$$0 = H(\infty) = \sum_j G_j(\frac{1}{\infty}) + C = \sum_j G_j(0) + C = \sum_j C(G_j) + C,$$

where $C(G_j)$ is the constant term of the polynomial G_j. Hence it follows that

$$C = -\sum_j C(G_j).$$